Aperiodic Sets of Square Tiles with Colored Corners

Ares Lagae *Jarkko Kari* *Philip Dutré*

Report CW460, August 2006

Katholieke Universiteit Leuven
Department of Computer Science
Celestijnenlaan 200A – B-3001 Heverlee (Belgium)

Aperiodic Sets of Square Tiles with Colored Corners

Ares Lagae *Jarkko Kari* *Philip Dutré*

Report CW460, August 2006

Department of Computer Science, K.U.Leuven

Abstract

In this paper we formalize the concept of square tiles with colored corners, a new kind of tiles closely related to Wang tiles. We construct aperiodic sets of square tiles with colored corners and several new aperiodic sets of Wang tiles using isomorphisms between tilings. The smallest aperiodic set of square tiles with colored corners we have created consists of 44 tiles over 6 colors.

Keywords : discrete mathematics, automata theory, aperiodic tilings, Wang tiles, corner tiles

Aperiodic Sets of Square Tiles with Colored Corners

Ares Lagae * Jarkko Kari † Philip Dutré *

Department of Computer Science, Katholieke Universiteit Leuven *

Department of Mathematics, University of Turku †

Abstract

In this paper we formalize the concept of square tiles with colored corners, a new kind of tiles closely related to Wang tiles. We construct aperiodic sets of square tiles with colored corners and several new aperiodic sets of Wang tiles using isomorphisms between tilings. The smallest aperiodic set of square tiles with colored corners we have created consists of 44 tiles over 6 colors.

1 Introduction

The work in this paper is motivated by recent developments in the field of computer graphics. In computer graphics, Wang tiles are often used [Sta97, HDK01, CSHD03, Wei04, LD05]. The tiles are usually filled with a signal that is complex or expensive to compute (e.g. textures, geometry or point distributions). Thereafter, arbitrary large non-periodic amounts of that signal can be generated very efficiently by generating a stochastic tiling.

Recently, we have shown that square tiles with colored corners, rather than Wang tiles, square tiles with colored edges, are better suited for these applications [LD06]. The main reason is that Wang tiles do not constrain their diagonal neighbors, and therefore cannot ensure continuity of the signal over tile corners. Corner tiles guarantee continuity with all their neighbors, and are not subject to this problem. Therefore, the signals are more easily reproduced using square tiles with colored corners.

To our knowledge, corners tiles have not been studied before in the tiling literature. In this paper we formalize the concept of square tiles with colored corners, and construct aperiodic sets of corner tiles based on isomorphisms

*{ares.lagae,philip.dutre}@cs.kuleuven.ac.be

†jkari@utu.fi

between Wang tilings and tilings of square tiles with colored corners. The smallest aperiodic set of corner tiles we have constructed consists of 44 tiles over 6 colors.

This paper is structured as follows. In section 2 we formally define square tiles with colored corners. Section 3 shows how to construct an aperiodic corner tile set from an aperiodic Wang tile set, and section 4 shows how to do the inverse. In section 5 we conclude.

2 Square Tiles with Colored Corners

Wang tiles are unit square tiles with colored edges. A Wang tile set is a finite set of Wang tiles. We consider tilings of the infinite Euclidean plane using arbitrarily many copies of the tiles in the given tile set. Tiles are placed on the integer lattice points of the plane, with their edges oriented horizontally and vertically. The tiles may not be rotated. The tiling is valid if everywhere the contiguous edges have matching colors.

Square tiles with colored corners, or corner tiles, are defined analogously. Like Wang tiles, corner tiles may not be rotated, and adjoining corners must have matching colors.

Let T be a finite set of tiles, and $f : \mathbb{Z}^2 \to T$ a tiling. Tiling f is periodic with period $(a, b) \in \mathbb{Z}^2 - \{(0, 0)\}$ iff $f(x, y) = f(x + a, y + b)$ for every $(x, y) \in \mathbb{Z}^2$. A tile set is called aperiodic iff there exists a valid tiling and there does not exist any periodic valid tiling.

In 1961, Wang conjectured that if a set of tiles can tile the plane, then they can always be arranged to do so periodically [Wan61, Wan65]. The conjecture was later refuted by Berger [Ber66] who constructed the first aperiodic set counting 20426 tiles. This number was subsequently greatly reduced. Currently, the smallest aperiodic Wang tile set consists of 13 tiles over 5 colors, and is due to [Cul96].

3 Constructing an Aperiodic Corner Tile Set from an Aperiodic Wang Tile Set

Because Wang tiles and corner tiles are so closely related, we construct aperiodic sets of corner tiles using isomorphisms between Wang tilings and corner tilings. In this section, we present five such construction methods: diagonal translation, horizontal translation, vertical translation, rotation and subdivision.

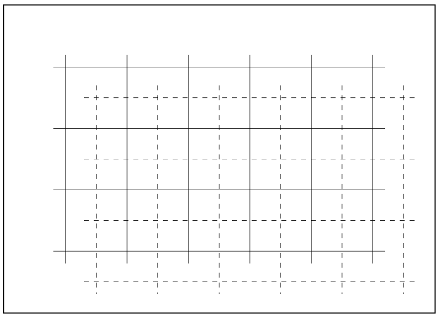


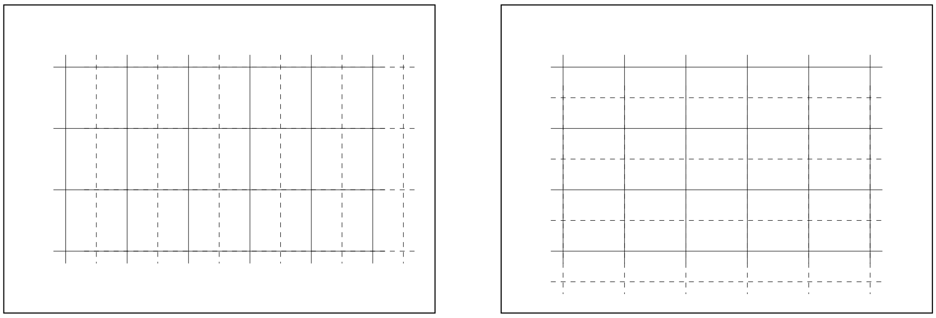
Figure 1: Diagonal translation. The lattice of the corner tiles (dashed lines) is translated diagonally with respect to the lattice of the Wang tiles (solid lines).

The aperiodic corner tile sets we construct are based on small aperiodic Wang tile sets. In this paper we use the following five aperiodic Wang tile sets (also see appendix A):

- the aperiodic set of 32 Wang tiles over 16 colors, obtained by cutting up a tiling by Penrose kites and darts [GS86, page 593];
- the aperiodic set of 24 Wang tiles over 24 colors, constructed from the Ammann prototiles (set A2) [GS86, page 593];
- the aperiodic set of 16 Wang tiles over 6 colors, generated using the Ammann set A2 and Ammann bars [GS86, page 595];
- the aperiodic set of 14 Wang tiles over 6 colors, constructed in 1996 by Kari [Kar96] with a method based on Mealy machines that multiply Beatty sequences of real numbers by rational constants; and
- the aperiodic set of 13 Wang tiles over 5 colors, created also in 1996 by Karel Culik II [Cul96] using a construction method based on the method developed by Kari.

3.1 Diagonal Translation

The diagonal translation construction method works for an arbitrary aperiodic set of Wang tiles. The corner tiles are placed on a lattice translated diagonally with respect to the lattice of the Wang tiles, as shown in figure 1. Each Wang tile is given a distinct color, and the edge colors of the Wang tiles are ignored. Each corner of each corner tile now receives the color of the Wang tile it lies on. The corner tile set constructed with this method consists of a single tile for each valid two by two square configuration of Wang tiles, and the number of colors used by the corner tile set equals the number of tiles in the Wang tile set. If the Wang tile set is aperiodic, then



(a) (b)

Figure 2: Horizontal and vertical translation. The lattice of the corner tiles (dashed lines) is translated (a) horizontally and (b) vertically with respect to the lattice of the Wang tiles (solid lines).

0	4	0	3	0	2	2	3	2	2	2	1	3	1	3	0	5	2	5	3	5	5
1	2	1	5	1	5	3	4	3	4	3	0	4	0	4	0	2	3	2	3	2	3
5	1	3	1	3	0	5	2	5	3	5	5	5	1	2	3	2	2	2	1	2	3
2	1	3	0	3	0	3	3	3	3	3	3	3	1	4	4	4	4	4	0	2	4
2	2	2	1	4	0	4	4	3	0	3	4	4	0	4	4	2	4	2	3	2	2
2	4	2	0	2	1	2	3	5	1	5	3	3	1	3	3	5	2	5	5	5	5
1	5	1	2	1	2	1	3	1	5	1	1	1	5	1	2	0	3	0	2	0	1
0	2	0	2	0	3	0	3	0	3	0	1	1	2	1	2	0	4	0	4	0	0

Figure 3: An aperiodic set of 44 corner tiles over 6 colors, constructed from the aperiodic set of 16 Wang tiles over 6 colors, using horizontal translation.

the corner tile set will clearly also be aperiodic. The second column of table 1 summarizes the results obtained with this construction method.

3.2 Horizontal and Vertical Translation

The horizontal and vertical translation construction methods are very similar to the diagonal translation construction method, but now the corner tiles are placed on a lattice translated horizontally or vertically with respect to the lattice of the Wang tiles, as shown in figure 2. A corner of a corner tile now receives the edge color of the edge of the Wang tile it lies on. In general, this is not an isomorphism, because all vertical or horizontal edges of the Wang tiles are ignored. However, for certain aperiodic Wang tile sets, this morphism is bijective. The third and fourth column of table 1 summarize the results obtained with this construction method. The aperiodic set of 44 corner tiles over 6 colors obtained using the horizontal translation method

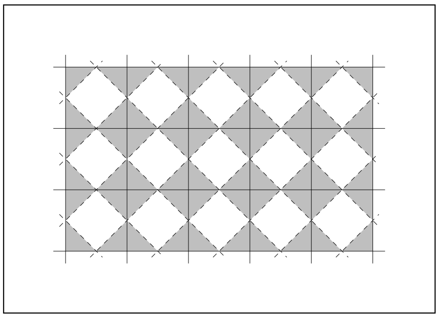


Figure 4: Rotation. The lattice of the corner tiles (dashed line) is rotated with respect to the lattice of the Wang tiles (solid line). This results in two kinds of tiles: black tiles and white tiles.

1	0
0	1

0	1
1	0

Figure 5: A white and a black corner tile that enforce a checkerboard pattern.

is shown in figure 3.

3.3 Rotation

The rotation construction method is somewhat more complicated than previous methods. The corner tiles are placed on a lattice rotated 45 degrees counterclockwise with respect to the lattice of the Wang tiles, as shown in figure 6. A corner of a corner tile receives the edge color of the edge of the Wang tile it lies on. However, the rotation results in two kinds of corner tiles: white tiles, corresponding to a single Wang tile, and black tiles, corresponding to a 2 by 2 square configuration of Wang tiles. The white and black tiles follow a checkerboard pattern. A corner tiling constructed this way cannot have a period that maps white tiles onto white tiles (or black tiles onto black tiles), because the Wang tiling is aperiodic, and a period that maps black tiles onto white tiles, or vice versa, is also impossible, because twice that period maps white tiles onto white tiles. We only need to enforce that all valid tilings with the corner tile set follow the checkerboard pattern. This can be done by superimposing on each corner tile one of the tiles in figure 5, effectively doubling (at most) the number of colors of the corner tile set. The fifth column of table 1 summarizes the results obtained with this construction method.

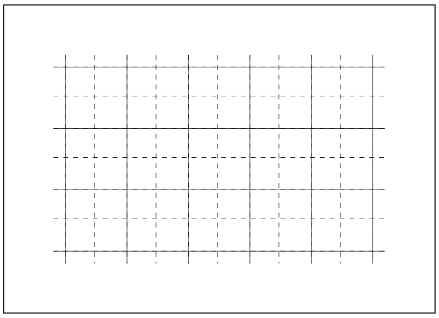


Figure 6: Subdivision. The lattice of the corner tiles (dashed line) is obtained by subdividing the lattice of the Wang tiles (solid line), such that each Wang tile corresponds with four corner tiles.

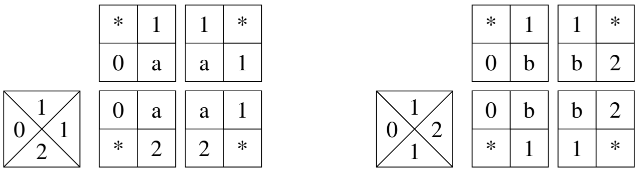


Figure 7: Two tiles from the aperiodic set of 13 Wang tiles over 5 colors, numbered a and b, and the corner tiles they produce. The star is a new color.

3.4 Subdivision

The final construction method we discuss is subdivision. The corner tiles are placed on a lattice obtained by subdividing the lattice of the Wang tiles, as shown in figure 6. Each Wang tile corresponds to four corner tiles. The corners that lie on the middle of an edge of a Wang tile receive the color of that edge. The corners that lie in the center of a Wang tile are colored with a color that uniquely determines that Wang tile. One additional color is assigned to the rest of the colors of all corner tiles. This procedure is illustrated in figure 7. The corner tile set obtained this way counts four times as much tiles as the Wang tile set. The number of colors is equal to the sum of the number of Wang tiles and the number of colors used in the Wang tile set plus one. If the Wang tile set is aperiodic, then the corner tile set will also be aperiodic. The sixth column of table 1 summarizes the results obtained with this construction method. Note that the additional color can be one of the edge colors used in the Wang tile set (but not one of the colors that uniquely determine the Wang tiles). Also note that some tiles can be eliminated by grouping certain colors that uniquely determine the Wang tiles. For example, one of the corner tiles in figure 7 is eliminated in by merging the colors a and b. We were able to reduce the number of tiles mentioned in table 1, but we have not succeeded in constructing a tile

Wang tile set	diagonal translation	horizontal translation	vertical translation	rotation	subdivision
13/5	125/13	failed	failed	60(13+47)/9	52/19
14/6	214/14	failed	86/6	94(14+80)/9	56/21
16/6	87/16	44/6	44/6	49(16+33)/12	64/23
24/24	203/24	62/12	72/12	67(24+43)/24	96/49
32/16	114/32	failed	failed	90(32+58)/28	128/49

Table 1: The size of aperiodic sets of corner tiles constructed with the diagonal, vertical, and horizontal translation method, and the rotation and subdivision method. For each tile set, the number of tiles and the number of colors is shown, separated by a slash. For the rotation method, the number of white and black tiles is also indicated.

Wang tile set	diagonal translation	horizontal translation	vertical translation	rotation	subdivision
13/5	125/50	failed	failed	60/21	52/36
14/6	214/86	failed	86/36	94/23	56/40
16/6	87/44	44/24	44/24	49/32	64/44
24/24	203/72	62/42	72/38	67/38	96/72
32/16	114/84	failed	failed	90/52	128/96

Table 2: The size of aperiodic sets of Wang tiles constructed from aperiodic sets of corner tiles. Each entry in this table is the number of tiles and number of colors of the aperiodic Wang tile set constructed from the corner tile set at the corresponding position in table 1.

set smaller than the one shown in figure 3.

4 Constructing an Aperiodic Wang Tile Set from an Aperiodic Corner Tile Set

Constructing an aperiodic Wang tile set from an aperiodic corner tile set is done by encoding each unique combination of two corner colors along a horizontal or vertical edge into a single new edge color. The number of tiles in the aperiodic Wang tile set constructed with this method is the same as the number of tiles in the corner tile set, and the number of colors is squared at most. Table 2 shows the size of the resulting aperiodic Wang tile sets for each of the aperiodic corner tile sets of table 1.

From an aperiodic Wang tile set created this way we can again construct an aperiodic corner tile set. Table 3 and appendix B show sequences of aperiodic tile sets constructed by repeated application of previous construction methods. Note that the number of tiles in the tile sets is not always an increasing sequence.

13/5	14/6	16/6	24/24	32/16
13/5	14/6	16/6	24/24	32/16
60/9	94/9	49/12	67/24	90/28
60/21	94/23	49/32	67/38	90/52
126/42	210/46	113/52	171/76	212/104
126/84	210/136	113/80	171/118	212/164
336/149	596/204	257/160	367/224	464/308
336/228	596/372	257/226	367/292	464/386
702/456	1306/744	545/416	801/584	972/772
702/592	1306/1074	545/478	801/702	972/892
1584/1130	3016/1976	1157/956	1625/1380	2020/1752
1584/1343	3016/2459	1157/1090	1625/1502	2020/1880
3222/2686	6314/4918	2373/2108	3341/3004	4134/3760
3222/3010	6314/5778	2373/2238	3341/3190	4134/3968
6780/5884	13468/11152	4881/4476	6685/6340	8418/7876
6780/6313	13468/12267	4881/4746	6685/6498	8418/8136
13626/12626	27534/24534	9873/9348	13513/12996	17032/16272
13626/13246	27534/26410	9873/9606	13513/13254	17032/16692
27864/26192	56760/51952	20013/19212	26985/26436	34364/33268
27864/26981	56760/54291	20013/19746	26985/26666	34364/33796
55782/53962	114730/108582	40237/39204	54213/53332	69130/67592
55782/55102	114730/112450	40237/39710	54213/53742	69130/68440
112692/109576	232836/223104	81001/79420	108293/107348	138862/136652
112692/111013	232836/227851	81001/80474	108293/107714	138862/137720
225378/222026	468086/455702	162409/160372	217025/215428	278540/275440
225378/224134	468086/463514	162409/161366	217025/216134	278540/277148
452880/446984	942896/923376	325861/322732	433729/432004	558288/553844
452880/449645	942896/932899	325861/324818	433729/432634	558288/555996
905598/899290	1890594/1865798	652517/648484	868285/865268	1118222/1111992
905598/903262	1890594/1881458	652517/650446	868285/866558	1118222/1115424
...	...	...	...	...

Table 3: The size of aperiodic sets of Wang tiles and corner tiles constructed by repeated application of the rotation method.

5 Conclusion

In this paper we have formalized the concept of square tiles with colored corners, and we have shown how to construct an aperiodic set of square tiles with colored corners from an aperiodic set of Wang tiles, and vice versa. We have constructed several new aperiodic sets of Wang tiles and corner tiles. The smallest set of corner tiles we have constructed consists of 44 tiles over 6 colors. We have shown that, if W and C are the cardinalities of the smallest aperiodic Wang tile set and the smallest aperiodic corner tile set, then $W \leq C \leq 4W$.

Acknowledgments

Ares Lagae is funded as a Research Assistant by the Fund of Scientific Research - Flanders, Belgium (Aspirant F.W.O.- Vlaanderen). Jarkko Kari is supported by the Academy of Finland Grant 54102.

References

[Ber66] R. Berger. The undecidability of the domino problem. *Memoirs American Mathematical Society*, 66(1-72), 1966.

[CSHD03] Michael F. Cohen, Jonathan Shade, Stefan Hiller, and Oliver Deussen. Wang tiles for image and texture generation. *ACM Transactions on Graphics*, pages 287–294, 2003.

[Cul96] Karel Culik, II. An aperiodic set of 13 Wang tiles. *Discrete Mathematics*, 160(1-3):245–251, 1996.

[GS86] B. Grünbaum and G. C. Shepard. *Tilings and Patterns.* W. H. Freeman and Company, 1986.

[HDK01] Stefan Hiller, Oliver Deussen, and Alexander Keller. Tiled blue noise samples. *Proceedings of Vision Modeling Visualization 2001*, pages 265–272, 2001.

[Kar96] Jarkko Kari. A small aperiodic set of Wang tiles. *Discrete Mathematics*, 160(1-3):259–264, 1996.

[LD05] Ares Lagae and Philip Dutré. A procedural object distribution function. *ACM Transactions on Graphics*, 24(4):1442–1461, 2005.

[LD06] Ares Lagae and Philip Dutré. An alternative for wang tiles: Colored edges versus colored corners. *Submitted to ACM Transactions on Graphics*, 2006.

[Sta97] J. Stam. Aperiodic texture mapping. Technical Report ERCIM-01/97-R046, European Research Consortium for Informatics and Mathematics (ECRIM), 1997.

[Wan61] H. Wang. Proving theorems by pattern recognition II. *Bell Systems Technical Journal*, 40:1–42, 1961.

[Wan65] H. Wang. Games, logic and computers. *Scientific American*, 213(5):98–106, 1965.

[Wei04] Li-Yi Wei. Tile-based texture mapping on graphics hardware. *Proceedings of the ACM SIGGRAPH/EUROGRAPHICS conference on Graphics hardware*, pages 55–63, 2004.

A Aperiodic Sets of Wang Tiles

This appendix shows the aperiodic sets of Wang tiles used in this paper. The colors of each tile set were relabeled to form an integer sequence starting with 0 and ending with the number of colors minus one.

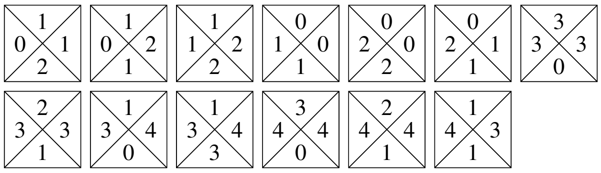


Figure 8: The aperiodic set of 13 Wang tiles over 4 colors.

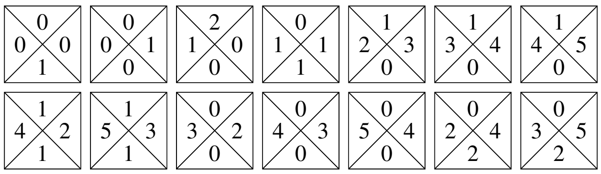


Figure 9: The aperiodic set of 14 Wang tiles over 6 colors.

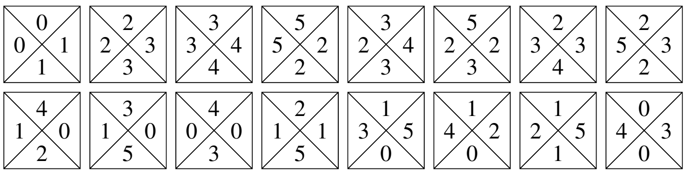


Figure 10: The aperiodic set of 16 Wang tiles over 6 colors.

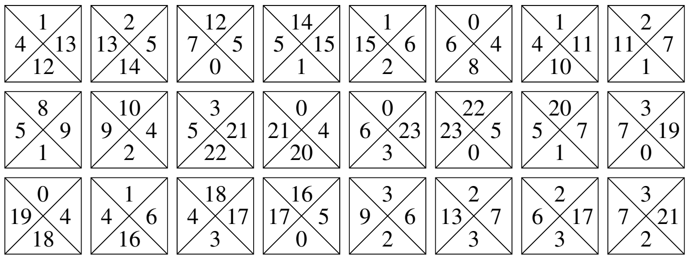


Figure 11: The aperiodic set of 24 Wang tiles over 24 colors.

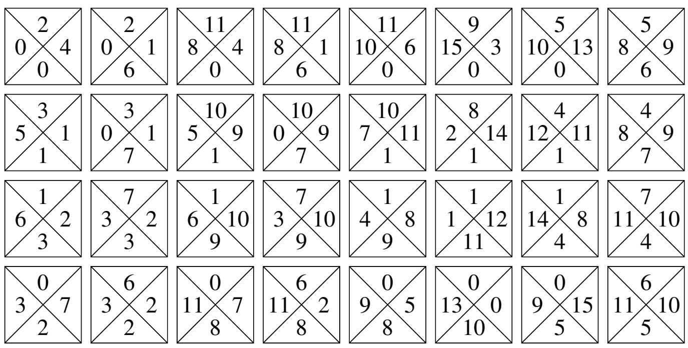


Figure 12: The aperiodic set of 32 Wang tiles over 16 colors.

B Sequences of Aperiodic Sets of Wang Tiles and Corner Tiles

This appendix shows the size of the tile set for sequences of aperiodic sets of Wang tiles and corner tiles generated by repeated application of the construction methods discussed in this paper.

13/5	14/6	16/6	24/24	32/16
13/5	14/6	16/6	24/24	32/16
125/13	214/14	87/16	203/24	114/32
125/50	214/86	87/44	203/72	114/84
1240/125	3052/214	286/87	540/203	280/114
1240/418	3052/1208	286/160	540/326	280/204
14810/1240	58910/3052	777/286	2068/540	662/280
14810/4523	58910/19274	777/464	2068/990	662/512
160168/14810	1213029/58910	1642/777	7092/2068	1124/662
160168/45400	1213029/358500	1642/1086	7092/3376	1124/924
3089844/160168	...	3599/1642	8910/7092	1904/1124
3089844/713351		3599/2222	8910/5598	1904/1606
...		5476/3599	38028/8910	3416/1904
		5476/3812	38028/18540	3416/2802
		10519/5476	104122/38028	4406/3416
		10519/7208	104122/53228	4406/3900
		17046/10519	80000/104122	7428/4406
		17046/10988	80000/49684	7428/5404
		21439/17046	481034/80000	9558/7428
		21439/16254	481034/242878	9558/8342
		43012/21439	227632/481034	12156/9558
		43012/29014	227632/135242	12156/10280
		45707/43012	1646934/227632	17172/12156
		45707/32056	1646934/788812	17172/14338
		68894/45707	2226936/1646934	21208/17172
		68894/51056	2226936/1020870	21208/18092
		114857/68894	...	33190/21208
		114857/76572		33190/28318
		106726/114857		40884/33190
		106726/83188		40884/34492
		200847/106726		45956/40884
		200847/140406		45956/41336
		216944/200847		52412/45956
		216944/145104		52412/43736
		255609/216944		62968/52412
		255609/205668		62968/58766
		474750/255609		86348/62968
		474750/331718		86348/66868
		335757/474750		103950/86348
		335757/251482		103950/85634
		540042/335757		116652/103950
		540042/392432		116652/103480
		739749/540042		140256/116652
		739749/492480		140256/122110
		...		...

Table 4: The size of aperiodic sets of Wang tiles and corner tiles constructed by repeated application of the diagonal translation method.

13/5	14/6	16/6	24/24	32/16
13/5	14/6	16/6	24/24	32/16
failed	failed	44/6	62/12	failed
		44/24	62/42	
		138/24	156/42	
		138/82	156/104	
		438/82	436/104	
		438/271	436/272	
		1406/271	1154/272	
		1406/854	1154/676	
		4572/854	3116/676	
		4572/2634	3116/1724	
		14997/2634	8354/1724	
		14997/7998	8354/4196	
		49532/7998	22472/4196	
		49532/24092	22472/10234	
		164425/24092	60344/10234	
		164425/72218	60344/24648	
		547868/72218	162170/24648	
		547868/215952	162170/60344	
		...	435680/59972	
			435680/162170	
			...	

Table 5: The size of aperiodic sets of Wang tiles and corner tiles constructed by repeated application of the horizontal translation method.

13/5	14/6	16/6	24/24	32/16
13/5	14/6	16/6	24/24	32/16
failed	86/6	44/6	72/12	failed
	86/36	44/24	72/38	
	504/36	138/24	192/38	
	504/216	138/82	192/122	
	2968/216	438/82	564/122	
	2968/1272	438/271	564/378	
	17464/1272	1406/271	1584/378	
	17464/7336	1406/854	1584/1110	
	102768/7336	4572/854	4548/1110	
	102768/41628	4572/2634	4548/3245	
	604736/41628	14997/2634	12912/3245	
	604736/233420	14997/7998	12912/9204	
	...	49532/7998	36864/9204	
		49532/24092	36864/26083	
		164425/24092	104952/26083	
		164425/72218	104952/73134	
		547868/72218	299220/73134	
		547868/215952	299220/204168	
		...	852480/204168	
			852480/567434	
			...	

Table 6: The size of aperiodic sets of Wang tiles and corner tiles constructed by repeated application of the vertical translation method.

13/5	14/6	16/6	24/24	32/16
13/5	14/6	16/6	24/24	32/16
60/9	94/9	49/12	67/24	90/28
60/21	94/23	49/32	67/38	90/52
126/42	210/46	113/52	171/76	212/104
126/84	210/136	113/80	171/118	212/164
336/149	596/204	257/160	367/224	464/308
336/228	596/372	257/226	367/292	464/386
702/456	1306/744	545/416	801/584	972/772
702/592	1306/1074	545/478	801/702	972/892
1584/1130	3016/1976	1157/956	1625/1380	2020/1752
1584/1343	3016/2459	1157/1090	1625/1502	2020/1880
3222/2686	6314/4918	2373/2108	3341/3004	4134/3760
3222/3010	6314/5778	2373/2238	3341/3190	4134/3968
6780/5884	13468/11152	4881/4476	6685/6340	8418/7876
6780/6313	13468/12267	4881/4746	6685/6498	8418/8136
13626/12626	27534/24534	9873/9348	13513/12996	17032/16272
13626/13246	27534/26410	9873/9606	13513/13254	17032/16692
27864/26192	56760/51952	20013/19212	26985/26436	34364/33268
27864/26981	56760/54291	20013/19746	26985/26666	34364/33796
55782/53962	114730/108582	40237/39204	54213/53332	69130/67592
55782/55102	114730/112450	40237/39710	54213/53742	69130/68440
112692/109576	232836/223104	81001/79420	108293/107348	138862/136652
112692/111013	232836/227851	81001/80474	108293/107714	138862/137720
225378/222026	468086/455702	162409/160372	217025/215428	278540/275440
225378/224134	468086/463514	162409/161366	217025/216134	278540/277148
452880/446984	942896/923376	325861/322732	433729/432004	558288/553844
452880/449645	942896/932899	325861/324818	433729/432634	558288/555996
905598/899290	1890594/1865798	652517/648484	868285/865268	1118222/1111992
905598/903262	1890594/1881458	652517/650446	868285/866558	1118222/1115424
...	...	...	...	...

Table 7: The size of aperiodic sets of Wang tiles and corner tiles constructed by repeated application of the rotation method.

13/5	14/6	16/6	24/24	32/16
13/5	14/6	16/6	24/24	32/16
52/19	56/21	64/23	96/49	128/49
52/36	56/40	64/44	96/72	128/96
208/89	224/97	256/109	384/169	512/225
208/176	224/192	256/216	384/336	512/448
832/385	896/417	1024/473	1536/721	2048/961
832/768	896/832	1024/944	1536/1440	2048/1920
3328/1601	3584/1729	4096/1969	6144/2977	8192/3969
3328/3200	3584/3456	4096/3936	6144/5952	8192/7936
13312/6529	14336/7041	16384/8033	24576/12097	32768/16129
13312/13056	14336/14080	16384/16064	24576/24192	32768/32256
53248/26369	57344/28417	65536/32449	98304/48769	131072/65025
53248/52736	57344/56832	65536/64896	98304/97536	131072/130048
212992/105985	229376/114177	262144/130433	393216/195841	524288/261121
212992/211968	229376/228352	262144/260864	393216/391680	524288/522240
851968/424961	917504/457729	1048576/523009	1572864/784897	2097152/1046529
851968/849920	917504/915456	1048576/1046016	...	...
...	...	...		

Table 8: The size of aperiodic sets of Wang tiles and corner tiles constructed by repeated application of the subdivision method.